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* Green's theorem :—

The theorem states that "If ϕ and ψ are two continuously differentable scalar function such that $\nabla \phi$ and $\nabla \psi$ are also continuosly differentable, then

$$\iiint (\phi \nabla \psi - \psi \nabla \phi)\, d\vec{s} = \iiint (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dV."$$

Proof :

Let a vector function $\vec{V}$ be given by $\vec{V} = \phi \nabla \psi$ ——— (1)

Where ϕ and ψ are scalar function so that $\vec{V}$ is product of a scalar function ϕ and the gradient of another scalar ψ.

Writing equation (1) in terms of the components we get

$$V_x = \phi \frac{\partial \psi}{\partial x}, \quad V_y = \phi \frac{\partial \psi}{\partial y}, \quad V_z = \phi \frac{\partial \psi}{\partial z}$$

$$\therefore \nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\phi \frac{d\psi}{dx}\right) + \frac{d}{dy}\left(\phi \frac{d\psi}{\partial y}\right) + \frac{d}{dz}\left(\phi \frac{d\psi}{\partial z}\right)$$

$$= \left\{\phi \frac{d^2\psi}{dx^2} + \frac{d\phi}{dx}\frac{\partial \psi}{dx}\right\} + \left\{\phi \frac{\partial^2 \psi}{dy^2} + \frac{d\phi}{dy}\frac{\partial \psi}{\partial y}\right\} +$$

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$$\left\{ \phi \frac{\partial^2 \psi}{\partial z^2} + \frac{d\phi}{dz} \frac{d\psi}{dz} \right\}$$

$$= \left\{ \phi \frac{\partial^2 \psi}{dx^2} + \phi \frac{\partial^2 \psi}{dy^2} + \phi \frac{\partial^2 \psi}{dz^2} \right\} + \left\{ \frac{d\phi}{dx} \frac{d\psi}{dx} + \frac{d\phi}{dy} \frac{d\psi}{dy} + \frac{d\phi}{dz} \frac{d\psi}{\partial z} \right\}$$

$$= \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \cdot \psi \quad \text{——— (2)}$$

By Gauss's theorem of divergence, we have

$$\iint_S \vec{V} \cdot \vec{n} \, ds = \iiint \cdot (\nabla \times \vec{V}) \cdot dv \quad \text{——— (3)}$$

Hence subetituing from equations (1), (2); (3) we get

$$\iint_S (\psi \nabla \phi) \cdot \vec{n} \, ds = \iiint_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) \, dv \quad \text{——— (4)}$$

If we interchange ϕ and ψ in equation (4), we get

$$\iint_S (\psi \nabla \phi \cdot \vec{n} \, ds) = \iiint_V (\psi \nabla^2 \phi + \nabla \phi \cdot \nabla \psi) \, dv. \quad \text{——— (5)}$$

Now subtracting equation (5) from (4), we get

$$\iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot \vec{n} \, ds = \iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dv$$

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or $\iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\vec{S} = \iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dv$ — (6)

This is green's theorem.

Now we derive Gauss's Theorem from Green's Theorem:

If we put $\phi = 1$, $\psi = \psi$ in equation (6), we have

$$\iiint_V \Delta^2 \psi \, dv = \iint_S (\text{grad}\, \psi) \cdot ds \quad \text{or,} \quad \iiint_V \text{div}(\text{grad}\, \psi)\, dv = \iint_S (\text{grad}\, \psi) \cdot ds$$

If we have a vector field $v = \text{grad}\, \psi$, then this equation leads to

$$\iiint \text{div}\, v \, dv = \iint v \cdot ds = \iint v \cdot n \, ds.$$

It is Gauss's theorem.